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Misconceptions and Semiotics: a comparison

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Abstract

Following D'Amore's constructive interpretation for the term misconception, we propose a semiotic approach to misconceptions, within the theoretical frameworks proposed by Raymond Duval and Luis Radford.

Introduction

In this article we deal with one of the most common terms for decades in Mathematics Education research, the word "*misconception*", interpreted according to a constructive perspective proposed by D'Amore (1999: p. 124): «A misconception is a wrong concept and therefore it is an event to avoid; but it must not be seen as a totally and certainly negative situation: we cannot exclude that to reach the construction of a concept, it is *necessary* to go through a temporary misconception that is being arranged». According to this choice, misconceptions are considered as steps the students must go through, that must be controlled under a didactic point of view and that are not an obstacle for students' future learning if they are bound to *weak and unstable* images of the concept; they represent, instead, an obstacle to learning if they are rooted in *strong and stable models*. For further investigation into this interpretation, look in D'Amore, Sbaragli (2005).

To understand what a misconception is, we believe it is necessary to make clear what is a concept and a conceptualization. Taking the special epistemological and ontological nature of mathematical objects as a starting point, we will show that mathematics requires a specific cognitive functioning that coincides with a complex semiotic activity immersed in systems of historical and cultural signification. This paper highlights that handling the semiotic activity is bristling with difficulties that hinder correct conceptual acquisition.

We will follow a constructive approach to misconceptions, analyzing them within the semiotic-cognitive and semiotic-cultural frameworks, upheld by Raymond Duval and Luis Radford respectively.

Theoretical framework

D'Amore's constructive approach to misconceptions

The problem of misconceptions developed within cognitive psychology studies, aiming at understanding the formation of concepts. In what follows, we refer to D'Amore (1999), but for the sake of brevity, we will not quote him.

This kind of approach focusses on the cognitive activity of the individual who is exposed to adequate stimuli and solicitations, and adapts his cognitive structures through assimilation and accommodation processes. The cognitive structures we mentioned above are characterised by two important functions that the human mind is able to perform: images and models formation.

The main characteristics of images and models are:

- Subjectivness, i.e. a strong relationship with individual experiences and characteristics.
- Absence of a proper sensorial productive input.
- Relation to a thought, therefore it does not exist per se, as a unique entity.
- Sensory and bound to senses.

An image is *weak* and transitory and accounts for the mathematical activity the pupil is exposed to in the learning process; it undergoes changes to adapt to more complex and rich mathematical situations set by didactical engineering as a path to reach a concept C.

A model has a *dynamical* character and it is seen as a limit image of successive adaptations to richer and richer mathematical situations. We recognise the limit image when a particular image doesn't need further modifications as it encounters new and more difficult situations. We can conclude that a model is a strong and *stable* image of the concept C the teacher wants the pupil to learn. A model among the images is the definitive one which contains the maximum of information and it is stable when facing many further solicitations. When an image is formed there are two possibilities:

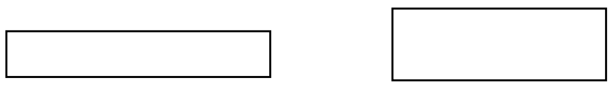
- The model M is the correct representation for the concept C.
- The model M is formed when the image is incomplete and it had to be further broadened. At this point it is more difficult to reach the concept C, because of the strength of M towards changes.

The adaptive process the student has to handle in his path towards the construction of a concept gives rise to a cognitive and emotional *conflict,* since he has to move to a new cognitive tool when the one he was using was working well; we usually call such conflict an error and the student requires specific support on the part of the teacher.

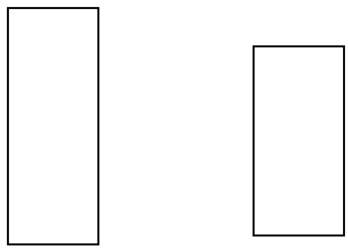
An image that worked well, has become inappropriate in a new situation and needs to be broadened for further use of the concept, is called *a misconception.* In the constructive perspective we have chosen, a misconception is not seen as a negative phenomenon, as long as it is bound to weak images. As we have already said, misconceptions are necessary stages the pupil has to go through in his learning process, and they must be controlled under a didactic point of view to ensure they are bound to modifiable images, and not to stable models that would hinder the student's conceptual acquisition.

We propose a classical primary school example of this path that leads the pupil towards the conceptualisation, starting from an image and ending with a model, passing through a cognitive conflict.

A grade 1 primary school student has always seen the drawing of a rectangle "lying" on its horizontal base with its height vertical and shorter. He constructed this image of the concept "rectangle" that has always been confirmed by experience. Most textbooks propose this prototypical image:



At a certain point the teacher proposes a different image of the rectangle that has the base smaller than its height.



The pupil's spontaneous denomination in order to adapt the concept already assumed is extremely meaningful: he defines this new shape as "standing rectangle", opposed to the former "lying rectangle", which expresses the more inclusive character of this image.

This denomination testifies the positive outcome of a cognitive conflict between a misconception (an improper fixed image of the concept "rectangle") and the new image wisely proposed by the teacher. The student already had an image bound to his embodied sensorial activity and the teacher's new proposal, obliging the student to move to a higher level of generality of this mathematical object.

An example of a misconception bound to a model that hinders the pupil's cognitive development is of a grade 11 high school pupil dealing with second degree equations.

We propose the solution in an assessment of the following equation: $2x^2+3x+5 = 0$

The student behaves as follows: $2x^2 = -3x-5$, $x = \pm\sqrt{(-3x-5)/2}$
At this point, he is unable to go further, even with the teacher's help. We highlight that the solution of second degree equations had already been explained to the class.

In this example, we can see how the procedure for the solution of first degree equations condensed into a strong model that didn't change even after the teacher's further explanations and mathematical activities.

This example shows that a misconception is not a lack of knowledge or a wrong concept, but knowledge that doesn't work in a broader situation.

In this purely psychological perspective, the construction of concepts in mathematics is independent of the semiotic activity. Signs are used only for appropriation and communication of the concept, *after* it has been obtained by other means. In mathematics, both when dealing with the production of new knowledge and with teaching-learning processes, this position is untenable, due to the ontological and epistemological nature of its objects.

In fact, we witness a reverse phenomenon: «Of course, we can always have the "feeling" that we perform treatments at the level of mental representations without explicitly mobilising semiotic representations. This introspective illusion is related to the lack of knowledge of a fundamental cultural and genetic fact: the development of mental representations is bound to the acquisition and interiorisation of semiotic systems and representations, starting with natural language» (Duval, 1995, p. 29).

Duval's semiotic-cognitive approach
Every mathematical concept refers to "non objects" that do not belong to our concrete experience; in mathematics ostensive referrals are impossible, therefore every mathematical concept intrinsically requires to work with semiotic representations, since we cannot display "objects" that are directly accessible.

The lack of *ostensive* referrals led Duval to assign the use of representations, organized in semiotic systems, a constitutive role in mathematical thinking; from this point of view he claims that there *isn't noetics without semiotics*. «The special epistemological situation of mathematics compared to other fields of knowledge leads to bestow upon semiotic representations a fundamental role. First of all they are the only way to access mathematical objects» (Duval, 2006).

The peculiar nature of mathematical objects allows outlining a specific cognitive functioning that characterises the evolution and the learning of mathematics. The cognitive processes that underlay mathematical practice are strictly bound to a complex semiotic activity that involves the coordination of at least two semiotic systems. We can say that conceptualisation itself, in Mathematics, can be identified with this complex coordination of several semiotic systems.

Semiotic systems are recognizable by:

- Organizing rules to combine or to assemble significant elements, for example letters, words, figural units.
- Elements that have a meaning only when opposed to or in relation with other elements (for example decimal numeration system) and by their use according to the organizing rules to designate objects (Duval, 2006).

Duval (1995a) identifies conceptualisation with the following cognitive-semiotic activities, specific for Mathematics:

- *formation* of the semiotic representation of the object, respecting the constraints of the semiotic system;
- *treatment* i.e. transformation of a representation into another representation in the same semiotic system;
- *conversion* i.e. the transformation of a representation into another representation in a new semiotic system.

The very combination of these three "actions" on a concept can be considered as the "construction of knowledge in mathematics"; but the coordination of these three actions is not spontaneous nor easily managed; this represents the cause for many difficulties in the learning of mathematics.

Duval bestows upon conversion a central role in the conceptual acquisition of mathematical objects:
«(…) registers coordination is the condition for the mastering of understanding since it is the condition for a real differentiation between mathematical objects and their representation. It is a threshold that changes the attitude towards an activity or a domain when it is overcome. (…) Now, in this coordination there is nothing spontaneous» (Duval, 1995b).

The coordination of semiotic systems, through the three cognitive activities mentioned above, broaden our cognitive possibilities because they allow transformations and operations on the mathematical object. When the object is accessible, distinguishing the representative from its representation and recognizing the common reference of several representations bound by semiotic transformations is guaranteed by the comparison between each single representation with the object. In Mathematics the situation is more complicated, because there is no object to carry out the distinction mentioned above and to guarantee the common reference of different representations to the object. The lack of ostensive referrals makes the semiotic activity problematic in terms of production, transformation and interpretation of signs.

From an educational point of view, this is a fundamental issue that leads the student to confuse the mathematical object with its representations and requires a conceptual acquisition of the object itself to govern the semiotic activity that in turn allows the development of mathematical knowledge. This self-referential situation is known as *Duval's cognitive paradox:* «(...) on one hand the learning of mathematical objects cannot be but a conceptual learning, on the other an activity on the objects is possible only through semiotic representations. This paradox can be for learning a true vicious circle. How could learners not confuse mathematical objects if they cannot have relationships but with semiotic representations? The impossibility of a direct access to mathematical objects, which can only take place through a semiotic representation leads to an *unavoidable* confusion. And, on the other hand, how can learners master mathematical procedures, necessarily bound to semiotic representations, if they do not already possess a conceptual learning of the represented objects?» (Duval, 1993, p. 38).

In the example that follows, given by Duval (2006) at the beginning of high school, we can see how the semiotic activity, in this case conversion, is crucial for the solution of the problem. Students encounter difficulties finding the solution because they are stuck on the fractional representation of rational numbers or, worse, they consider fractions and decimal representation different numbers. The mathematical procedure is grounded on the cognitive semiotic activity. The mathematics involved is very simple but the semiotic task is not trivial.

$1+1/2+1/4+1/5+\ldots = 2$

The following conversion solves the problem brilliantly, shifting from the fraction representation of rational numbers to the decimal one.

$1+0.5+0.25+0.20+\ldots = 2$

Radfrod's semiotic-cultural approach

Within the semiotic path we follow to understand mathematical thinking, we make a step forward and move on to Radford's semiotic- cultural framework.

Radford's theory of knowledge objectification, considers thinking a *mediated reflection* that takes place in accordance with the mode or form of *individuals' activity* (Radford, 2005):

- The *reflexive* nature refers to the relationship between the individual consciousness and a culturally constructed reality.
- The *mediated* nature refers to the means that orient thinking and allows consciousness to become aware of and understand the cultural reality; Radford calls such means *Semiotic Means of Objectification* (Radford, 2002). The word semiotic is used in a broader sense to include the whole of the individuals embodied experience that develops in terms of bodily actions, use of artifacts and symbolic activity: artifacts, gestures, deictic and generative use of natural language, kinaesthetic activity, feelings, sensations and Duval's semiotic systems. Semiotic Means of Objectification mustn't be

considered as practical and neutral technical tools, but they incarnate historically constituted knowledge. They bare the culture in which they have been developed and used. The semiotic means determine the way we interpret and understand reality that is given through our senses. The mediated nature of thinking is constitutive of our cognitive capabilities and makes thinking culturally dependent.

- *Activity* refers to the fact that mediated reflection is not considered here a solitary purely mental process, but it involves shared practices that the cultural and social environment considers relevant.

Before analyzing the learning process, we need to deal with the notion of mathematical object in Radford's objectification theory. Going beyond realist and empiristic ontologies, the theory of knowledge objectification considers mathematical objects culturally and historically generated by the mathematical activity of individuals. In agreement with the mediated reflexive nature of thinking and from the viewpoint of an anthropological epistemology Radford claims that «(…) Mathematical objects are fixed patterns of activity embedded in the always changing realm of reflective and mediated social practice» (Radford, 2004; p.21).

Learning is an objectification process that allows the pupil to become aware of the mathematical object that is culturally already there, but it is not evident to the student. Ontogenetically speaking, the student carries out a reflection on reality, not to construct and generate the object as it happens phylogentically, but to make sense of it. Learning is therefore an *objectification* process that transforms *conceptual and cultural* objects into objects of our *consciousness*. In this meaning-making process, the semiotic means of objectification within socially shared practices allow the student's individual space-time experience to encounter the general disembodied cultural object.

The access to the object and its conceptualization is only possible within a semiotic process and it is forged out of the multifaceted dialectical interplay of various semiotic means, with their range of possibilities and limitations. This multifaceted interplay synchronically involves, within reflexive activity, bodily actions, artefacts, language and symbols. At different levels of generality these three elements are always present. For example, at the first stage of generalization in algebra students have mainly recourse to gestures and deictic use of natural language, whereas in dealing with calculus the use of formal symbolism will be predominant, nevertheless without disregarding the kinaesthetic activity or the use of artefacts.

In the objectification process the student lives a conflict between his reflexive activity situated in his personal space-time embodied experience and the disembodied meaning of the general and ideal cultural object. The teaching-learning process has to face the dichotomy between the phylogenesis of the mathematical object and the ontogenesis of the learning process. The cognitive processes phylogenetically and ontogenetically involve the same reflexive activity, but with a significant difference: in the first case the mathematical object emerges as a fixed pattern; in the second case the object has its independent existence and the didactic engineering has to devise specific practices to allow the student becoming aware of such object.

To heal the conflict between embodied and disembodied meaning, the student has to handle more complex and advanced forms of representation «that require a kind of *rupture* with the ostensive gestures and contextually based key linguistic terms that underpin presymbolic generalizations» (Radford, 2003: p. 37).

The following example proposed by Radford (2005) shows the difficulty students encounter when they have to use algebraic symbolism that cannot directly incorporate their bodily experience. Students were asked to find the number of toothpicks for the n-th figure of the following sequence.

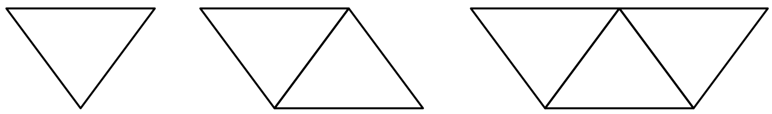


Fig. 1 *Fig. 2* *Fig. 3*

After resorting to gestures, deictic use of natural language, students manage to write the algebraic expression n+(n+1) [n is the number of the figure in the sequence], but they are not ready to carry out the trivial algebraic transformation that leads to 2n+1. The parentheses have a strong power in relating the algebraic representation to their visual and spatial designation of the figure, disregarding them implies a disembodiment of meaning that it is not easily accepted. Even though 2n+1 is synthactically equivalent to n+(n+1), the former expression requires a rupture with spatial based semiotic means of objectification and a leap to higher levels of generality.

Misconceptions: a semiotic interpretation

The semiotic approach we have outlined in the previous sections provides powerful tools to understand the nature of misconceptions. From what we said, the path that from weak images leads to strong models can be seen as the interiorisation of a complex semiotic activity; the student has acquired a correct *model* of the concept when he masters the coordination of a *set of representations*, relative to that concept, that is *stable* and effective in facing diverse mathematical situations. The student acquires control of an adequate set of representations, through an adaptation process that enlarges the representations of the set and coordinates them in terms of semiotic activity. From a semiotic point of view an i*mage*, is a *temporary* set of representations that needs to be developed, both in terms of representations and of their coordination, as the student faces new and more exhaustive solicitations.

A misconception is a set of representations that worked well in previous situations but it is inappropriate in a new one. If a misconception is relative to a weak image the student is able to enlarge the set of representations and he is also ready to carry out more complex semiotic operations. In this case a misconception is a necessary and useful step the student must go through. If, instead, a misconception is related to a strong model the student will refuse to incorporate new representations and commit himself to more elaborated semiotic transformations. At this point, the pupil's cognitive functioning is stuck and he is unable to solve problems, deal with non standard mathematical situations and broaden his conceptual horizon. His reasoning is bridled in repetitive cognitive paths related to the same representations and transformations. In this case, a misconception is a negative event that must be avoided.

The representations we mentioned above are Radford's Semiotic Means of Objectification, including also Duval's semiotic systems. We can broaden D'Amore's (2003, p.55-56) *constructivist* view point of mathematical knowledge based on Duval's semiotic operations (formation, treatment and conversion) on semiotic systems, to include also bodily activity and artefacts and deal with more general Semiotic Means of Objectification. The positive outcome of the construction of a mathematical concept is therefore the dialectical interplay of Semiotic Means of Objectification that includes also treatments and conversions on semiotic systems. Such positive outcome is not a plain solitary process but it is culturally embedded in shared activity and it must overcome three synchronically entangled turning points that give rise to misconceptions; for sake of clarity we will discuss them separately but to show how entangled they are we will propose always the same example to explicit them.

- The first turning point we discuss is the *cognitive paradox*. The first and only possible approach to the mathematical object the student has is with a particular semiotic means of objectification. It can be an artefact, a drawing or a linguistic expression. He necessarily identifies the object with the first representation he encounters and connecting it with others is not spontaneous and requires a specific didactic action to go through this misconception. The student spontaneously sticks to the first representation that worked well in the situation devised by the teacher, but he is in trouble when a new situation requires to connect the first representation to a new one, because he believes that such representation *is* the mathematical object.

We can take the prototypical example of the rectangle we analyzed in section 2.1. In primary school the first access to the rectangle usually is a drawing with the base longer than the height. The student thinks that the object rectangle is *that* drawing with *those* specific perceptual characteristics. He is in trouble when the teacher proposes the new representation; he calls it "standing rectangle". If the teacher hadn't exposed the student to a new solicitation that first misconception would have condensed into a model, hindering the pupil's further cognitive development.

- The second turning point is the *coordination* of a variety of representations. In terms of Semiotic Means of Objectification the student has to handle a very complicated situation. First of all, the semiotic means can be very different from each other in terms both of their characteristics and the way they are employed. For instance a gesture is very different from an algebraic expression. The first one is used spontaneously, while the second is submitted to strict syntactic rules. The first one is related to the kinaesthetic activity, whereas the second one is a semiotic system that does not incorporate the students' kinaesthetic experience in a direct manner. An algebraic expression requires treatment and conversion transformations, while these operations are impossible with gestures. The student has to handle a semiotic complexity that leads to misconceptions mainly related to the coordination of semiotic means. The interplay of heterogynous Semiotic Means of Objectification is not spontaneous and it requires a specific didactic action.

Let us turn back to the example of the rectangle. In his cognitive history the student will have to coordinate more and more representations of this object. We have seen that he started with a very simple drawing, perceptively effective. The teacher proposes a treatment that leads the student to consider a new representation that is in conflict with the previous one. This is not enough to construct a model of the rectangle. As the mathematical problems become more complicated he will need to resort, through conversions, to other semiotic systems like natural language, the cartesain system or the algebraic one. We can ask him if a square is a rectangle, at this point he needs to combine his perceptual experience bound to the figural semiotic system with the definition given in natural language. Many students cannot accept that a square is a rectangle. In high school we could ask him to calculate the area of a rectangle obtained by the intersection of four straight lines given as first degree equations. Although the problem is simple from a mathematical point of view, it puzzles the student because of a complex semiotic activity that involves conversions between cartesian and algebraic representations. In this case, conversion is a heavy task to accomplish because of non congruence phenomena (Duval, 2005a, pp. 55-59). The student has to face a misconception that will cause a compartmentalization of semiotic systems, hindering his semiotic degree of freedom.

The coordination of many representations is a source of misconception, also because, as recent researches in the field conducted by (D'Amore, 2006) show, semiotic transformations change the sense of mathematical objects. For the student each representation has its own meaning related to the nature of the semiotic means of objectification and to the shared practices on the object carried out through such representation. The misconception of the rectangle is a good example of this phenomenon. Students bestow different senses upon each representation, at such a point that the child calls them "lying" and "standing" rectangles, as if they were different objects. It turns out that keeping the same denotation of different representations is a cognitive objective difficult to acquire because it demands to handle many representations without accessing what is represented.

- The last turning point we want to discuss regards the *disembodiment of meaning*. We have seen that there is a dichotomy between the space-time situated embodied experience of the pupil and the disembodied general mathematical object. The student lives a conflict between the embodied and situated nature of his personal learning experience and the disembodied general nature of the mathematical object. The mathematical cognitive activity of the child cannot start but in an embodied manner resorting mainly to Semiotic Means of Objectification related to bodily actions and the use of artefacts. But, when the mathematical activity requires a higher level of generality, the student must also engage in abstract symbols; the toothpicks shows how difficult it is for the student to give up his space situated experience, and how the algebraic language is meaningful to him as long as it describes his contextual activity. The conflict between situated experience and the generality and abstraction of the mathematical object is a source of misconceptions. At present, it is not completely clear how the disembodiment of meaning takes place. We know that the disembodiment of meaning requires the coordination of Duval's semiotic systems, in terms of treatment and conversion, and what we usually do is to expose students to an abstract symbolic activity, aware that we must handle the rise of misconceptions. Turning back to the example of the rectangle, the "lying" rectangle and the "standing" one are symptoms of the embodied meaning bound to the student's perceptual and sensorial experience. The treatment between the two figurative representations implies a disembodiment of meaning that must continue as natural language and other semiotic systems will be introduced so that the pupil can grasp the general and abstract sense of the rectangle.

We have presented a thorough analysis of misconceptions from a semiotic perspective. Anyway, it is possible to single out from what we have said a pivot upon which the issue of conceptualization and misconception turns, i.e. the lack of ostensive referrals of mathematical objects. The inaccessibility of mathematical objects both imposes the use semiotic representations and makes the semiotic activity intrinsically problematic.

A first classification of misconceptions

From what we have said above, on the one hand it seems that misconceptions are somehow a necessary element of the learning of mathematics and on the other the role of the teacher is crucial to overcome them by supporting the student's ability to handle the semiotic activity. We have, hence, divided misconceptions into two big categories: *"unavoidable"* and *"avoidable"* (Sbaragli, 2005); the first *does not depend directly on the teacher's didactic transposition*, whereas the second *depends exactly on the didactic choices and didactic engineering devised by the teacher.* Avoidable misconceptions derive directly from teachers' choices and improper habits proposed to pupils by didactic praxis. Unavoidable misconceptions derive only *indirectly from teachers choices* and are bound to the need of beginning from a starting knowledge that, in general is not exhaustive of the whole mathematical concept we want to present.

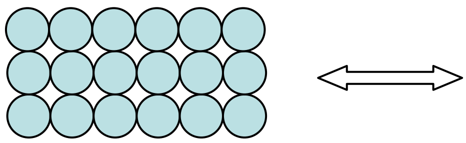
We will analyze avoidable and unavoidable misconceptions referring to the three turning points mentioned above.

"Unavoidableness"

"Unavoidable" misconceptions, that do not derive from didactical transposition and didactic engineering, depend mainly on the intrinsic unapproachableness of mathematical objects. Duval's (1993) paradox is a source of misconceptions that gives rise to an *unavoidable* confusion between semiotic representations and the object itself, especially when the concept is proposed for the first time. Another source of unavoidable misconceptions derives from the conflict between embodied and disembodied meaning of the mathematical concept. When the student learns a new mathematical concept he cannot begin to approach it with Semiotic means of Objectification related to his practical sensory-motor intelligence. These Semiotic Means of Objectification can lead the student to consider relevant "parasitical information", in contrast with the generality of the concept, bound to the specific representation and the perceptive and motor factors involved in his mathematical activity. The student *unavoidably* misses the generality of the mathematical object and grounds his learning only on his sensual experience.

The following example highlights an unavoidable misconception.

We know from literature (D'Amore and Sbaragli, 2005) that a typical misconception, rooted in the learning of natural numbers is that the product of two numbers is always greater than its factors. When students pass to the multiplication in Q, they do not accept that the product of two numbers can be smaller than its factors. They are stuck to the misconception that "multiplication always increases". This is true in N and it is reinforced by the embodied meaning enhanced by the array model of multiplication perceptually very strong, effective in the first stages of students' learning of arithmetic and in strong agreement with the idea of multiplication as a repeated sum. We can see that there is strong congruence between the figural representation and the symbolic one that makes conversion very natural.



$6\times3=6+6+6$

When we pass to Q and consider 6×0.2 what does it mean to sum 6, 0.2 times, and what is an array with 0.2 rows and 6 columns?

We can see how the strong identification of the mathematical object with its representation hinders the development of the concept, and it is also clear that this identification is an unavoidable passage.
This example clearly shows, on the one hand, the rupture that leads from embodied to disembodied general meaning, the student has to go through when he faces rational numbers and how difficult it is to give up the perceptual and sensory evocative power of the array. On the other hand, it is also evident that we cannot avoid the embodied meaning skipping directly to a general and formal definition of multiplication.

The array is an effective Semiotic Means of Objectification when the student *begins* to learn multiplication in N, but if there is no *specific didactic action* that fosters the generalizing process towards the mathematical concept, it condensates into a strong model, difficult to uproot. The array image of multiplication is a typical example of a parasitical model. This last remark opens the way for the discussion of avoidable misconceptions.

"Avoidableness"
Avoidable misconceptions derive directly from *didactic transposition and didactic engineering*, since they are a direct consequence of the teachers' choices.

We have seen that the cognitive paradox and disembodiment of meaning give rise to unavoidable misconceptions. Nevertheless the teacher has an important degree of freedom to intervene in the students' ability to handle the semiotic activity. Even if misconceptions are unavoidable they must be related to images without becoming stable models. This is possible if the student is supported in handling the complex semiotic activity, within socially shared practices, that fosters the *cognitive rupture*, allowing the pupil to incorporate his kinaesthetic experience in more complex and abstract semiotic means. The student thus goes beyond the embodied meaning of the object and endows it with its cultural interpersonal value. In this perspective, Duval (1995) offers important didactic indications to manage the rupture described above, when he highlights the importance of exposing the student, in a critical and aware manner, to many representations in different semiotic registers, overcoming also the cognitive paradox. Nevertheless didactic praxis is "undermined" by improper habits that expose pupils to univocal and inadequate semiotic representations, transforming avoidable misconceptions in strong models or giving rise to new ones.

An emblematic example of an inadequate semiotic choice that brings to improper and misleading information relative to the proposed concept, regards the habit of indicating the angle with a "little arc" between the two half-lines that determine it. Indeed, the limitedness of the "little arc" is in contrast with the boundlessness of the angle as a mathematical abstract "object". This implies that in a research involving students of the Faculty of Education, most of the persons interviewed claimed that the angle corresponds to the length of the little arc or to the limited part of the plane that it identifies, falling into an embarrassing contradiction; two half lines starting from a common point determine infinite angles! (Sbaragli, 2005).

An inadequate didactical transposition or didactic engineering can in fact strengthen the confusion, lived by the student, between the symbolic representations and the mathematical object. The result is that «the student is unaware that he is learning signs that stand for concepts and that he should instead learn concepts; if the teacher has never thought over this issue, he will believe that the student is learning concepts, while in fact he is only "learning" to use signs» (D'Amore, 2003; p. 43).

It thus emerges how often the choice of the representation, is not an aware didactical choice but it derives from teachers' wrong models. And yet, in order to avoid creating strong misunderstandings it is first required that the teacher knows the "institutional" meaning of the mathematical object that she wants her students to learn, secondly she must direct the didactical methods in a critical and aware manner.

From a didactical point of view, it is therefore absolutely necessary to overcome "unavoidable" misconceptions and prevent the "avoidable" ones, with particular attention to the Semiotic Means of Objectification, providing a great variety of representations appropriately organized and integrated into a social system of meaning production, in which students experience shared mathematical practices.

From what we have said, learning turns out to be a constructive semiotic process that entangles representations and concepts in a complex network, with the rise of misconceptions. Therefore the task of the teacher is to be extremely sensible towards misconceptions that can come out during the teaching-learning process. The teacher must be aware that what the student thinks as a correct concept, it can be a misconception rooted in an improper semiotic activity.

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